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A New Family of Relative Difference Sets in 2-Groups

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Dedicated to the memory of E. F. Assmus

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Abstract. We recursively construct a new family of $(2^{6d+4}, 8, 2^{6d+4}, 2^{6d+1})$ semi-regular relative difference sets in abelian groups G relative to an elementary abelian subgroup U . The initial case $d = 0$ of the recursion comprises examples of $(16, 8, 16, 2)$ relative difference sets for four distinct pairs (G, U) .

Keywords: relative difference set, recursive, building set, character

1. The square root problem

Let G be a group of order mu and U a normal subgroup of G of order u . If R is a k -subset of G then R is a (m, u, k, λ) relative difference set (RDS) in G relative to U provided that the multiset of differences rr'^{-1} for $r, r' \in R, r \neq r'$, contains every element of $G \setminus U$ exactly λ times and contains no element of U . If $k = u\lambda$ then the RDS is called *semi-regular* and the parameters are $(u\lambda, u, u\lambda, \lambda)$. In this paper we consider semi-regular RDSs with parameters of the form

$$(2^a, 2^b, 2^a, 2^{a-b}). \quad (1)$$

Several families of such RDSs have been constructed for $b \leq a/2$ [3]. However for $b > a/2$ the only known existence results for abelian groups are as follows:

THEOREM 1 *There is a $(2^a, 2^b, 2^a, 2^{a-b})$ RDS in the group $\mathbb{Z}_4^b \times \mathbb{Z}_2^{a-b}$, relative to the subgroup $U \cong \mathbb{Z}_2^b$ contained in \mathbb{Z}_4^b , for each b satisfying $a/2 < b \leq a$.*

THEOREM 2 *There is a $(2^{2b-1}, 2^b, 2^{2b-1}, 2^{b-1})$ RDS in any abelian group G of order 2^{3b-1} and exponent 4 relative to $U \cong \mathbb{Z}_2^b$, where U is contained within a subgroup of G isomorphic to \mathbb{Z}_4^b , for each odd $b \geq 1$.*

Theorem 1 is due to Jungnickel [6] (taking into account the well-known method of contraction [7]). Theorem 2 is due to Chen, Ray-Chaudhuri and Xiang [2]. Ganley [5] has shown that when $b = a$ the only abelian group G containing an RDS with parameters (1) is \mathbb{Z}_4^a , and Schmidt [9] has given further nonexistence results for $b > a/2$. Nonetheless

there is a large gap of understanding between the known existence and nonexistence results when $b > a/2$. We refer to this gap as the “square root problem” because it corresponds to the parameter relationship $u > \sqrt{k}$. In this section we give new solutions to the square root problem by exhibiting a $(16, 8, 16, 2)$ RDS for four distinct pairs (G, U) .

Relative difference sets are often studied in the context of a group ring $\mathbb{Z}[G]$ and group characters. The definition of a RDS immediately yields the group ring equation $RR^{(-1)} = k1_G + \lambda(G - U)$, where we identify R , $R^{(-1)}$ and G with the respective group ring elements $R = \sum_{r \in R} r$, $R^{(-1)} = \sum_{r \in R} r^{-1}$ and $G = \sum_{g \in G} g$. Characters of an abelian group G are homomorphisms from G to the multiplicative group of complex roots of unity, and we extend this homomorphism to the entire group ring in the natural way. The element R of $\mathbb{Z}[G]$ then satisfies the definition of a semi-regular RDS if and only if two conditions hold [7]: first, any character that is nonprincipal (i.e., nontrivial) on the subgroup U has a character sum over R of modulus $\sqrt{u\lambda}$ and second, any character that is principal (i.e., trivial) on the subgroup U but nonprincipal on the group G has a character sum of 0 over R .

Davis and Jedwab [3] describe a theoretical framework for constructing RDSs a piece at a time. We define a (a, m, t) building set (BS) on an abelian group G relative to a subgroup U to be a collection of t subsets of G (called building blocks), each of size a , such that for any nonprincipal character χ of G :

- (i) Exactly one building block has a character sum of modulus m and all other building blocks have character sum 0 if χ is nonprincipal on U and
- (ii) All building blocks have character sum 0 if χ is principal on U .

THEOREM 3 ([3], THEOREM 2.2) *Suppose there exists a (a, \sqrt{at}, t) BS $\{B_1, B_2, \dots, B_t\}$ on an abelian group G relative to a subgroup U of order u , where $at > 1$. Then $\cup_{i=1}^t g'_i B_i$ is a $(at, u, at, at/u)$ semi-regular RDS in G' relative to U , where G' is any abelian group containing G as a subgroup of index t and the g'_i lie in distinct cosets of G in G' .*

All the new RDSs of this paper arise from the following example.

Example 4. Let G be the group $\langle x, y, z, w | x^4 = y^4 = z^2 = w^2 = 1 \rangle \cong \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$ and let U be the subgroup $\langle x^2, y^2, w \rangle \cong \mathbb{Z}_2^3$. The subsets $B_1 = 1 + x + y + xyw + z(1 + x^3 + y^3 + x^3y^3w)$ and $B_2 = 1 + xy^2 + x^2yw + x^3y^3 + y^2zw(1 + x^3y^2 + x^2y^3w + xy)$ form a $(8, 4, 2)$ BS on G relative to U .

By Theorem 3 this implies there is a $(16, 8, 16, 2)$ RDS R in G' relative to U as follows:

1. $G' = \langle x'^8 = y^4 = z^2 = w^2 = 1 \rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$; $U = \langle x'^4, y^2, w \rangle$; $R = B_1 \cup x' B_2$.
2. $G' = \langle x^4 = y^4 = z'^4 = w^2 = 1 \rangle \cong \mathbb{Z}_4^3 \times \mathbb{Z}_2$; $U = \langle x^2, y^2, w \rangle$; $R = B_1 \cup z' B_2$.
3. $G' = \langle x^4 = y^4 = z^2 = w'^4 = 1 \rangle \cong \mathbb{Z}_4^3 \times \mathbb{Z}_2$; $U = \langle x^2, y^2, w'^2 \rangle$; $R = B_1 \cup w' B_2$.
4. $G' = \langle x^4 = y^4 = z^2 = w^2 = v'^2 = 1 \rangle \cong \mathbb{Z}_4^2 \times \mathbb{Z}_2^3$; $U = \langle x^2, y^2, w \rangle$; $R = B_1 \cup v' B_2$.

The following Mathematica commands can be used to verify that the building blocks of Example 4 satisfy the definition of a $(8, 4, 2)$ BS:

```
B1[x',y',z',w']:=1 + x + y + x y w + z (1 + x^3 + y^3
+ x^3 y^3 w);
B2[x',y',z',w']:=1 + x y^2 + x^2 y w + x^3 y^3 +
y^2 z w (1 + x^3 y^2 + x^2 y^3 w + x y);
Do[Print[i,j,k,l,B1[I^i,I^j,(-1)^k,(-1)^l],B2[I^i,I^j,(-1)^k,
(-1)^l]],{i,0,3},{j,0,3},{k,0,1},{l,0,1}]
```

The evaluation of B_1 and B_2 in the Do loop runs through all the possible character values. The output indicates that exactly one of the two blocks has character sum of modulus 4 for the appropriate characters, and that they both have character sum 0 for the other characters (the first character that prints out is the principal character, and that has a sum of 8 for both characters).

The quotient group $G/\langle w \rangle$ in Example 4 is isomorphic to $\mathbb{Z}_4^2 \times \mathbb{Z}_2$ and under this contraction the building blocks B_1 and B_2 are similar to the building blocks of the Arasu-Sehgal example [1]. In other words, the building blocks B_1 and B_2 can be viewed as “lifts” of the Arasu-Sehgal building blocks. This observation, together with a better understanding of the structure of B_1 and B_2 , might lead to a generalisation to higher order groups that would give further solutions to the square root problem.

2. A new family of semi-regular RDSs

In this section we use Example 4 as an initial case to construct recursively a new family of BSs and then, using Theorem 3, to obtain a new family of semi-regular RDSs. (For a summary of the current state of knowledge for semi-regular RDSs in abelian groups relative to an elementary abelian subgroup see [3] and [4].) The recursive construction of BSs follows the method of [3] in making use of the $p^r + 1$ hyperplanes of the group \mathbb{Z}_p^{2r} , regarded as a vector space of dimension 2 over $\text{GF}(p^r)$.

THEOREM 5 ([3], THEOREM 4.3) *Let G be an abelian group of order $p^{2r}a$ containing a subgroup $Q \cong \mathbb{Z}_p^{2r}$, where p is prime. Let H_0, H_1, \dots, H_{p^r} be the subgroups of G of order p^r corresponding to hyperplanes when viewed as subgroups of Q . Suppose there exists a (a, \sqrt{at}, t) BS on G/H_i relative to Q/H_i for each $i = 1, 2, \dots, p^r$. Then there exists a $(p^r a, p^r \sqrt{at}, p^r t)$ BS on G relative to H_0 .*

To apply Theorem 5 effectively we require information about the form of the quotient groups G/H_i and Q/H_i . We know (see Lemma 7 below) that if G has rank exactly $2r$ then by an appropriate choice of generators exactly r direct factors of G retain the same exponent in G/H_i (these are the direct factors which contain Q/H_i), whereas r have their exponent reduced by a factor of p . However Example 4 has a feature not previously considered: the subgroup U is contained in a subgroup of G isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2$ but not in a subgroup isomorphic to \mathbb{Z}_4^3 . To deal with this feature we begin with a group theoretic lemma.

LEMMA 6 *Let y_1, y_2, \dots, y_r be elements of an abelian group G and let H be a subgroup of G . If $\langle y_u \rangle \cap \langle y_j \mid j \neq u \rangle = \{1\}$ for each u in the range $1 \leq u \leq r$ and no nonidentity element of the form $\prod_{u=1}^r y_u^{j_u}$ is contained in H , then $\langle y_1 H, y_2 H, \dots, y_r H \rangle \cong \langle y_1 H \rangle \times \langle y_2 H \rangle \times \dots \times \langle y_r H \rangle$.*

Proof. We prove this by induction on r starting with the case $r = 2$. We claim that $\langle y_1 H \rangle \cap \langle y_2 H \rangle = \{H\}$. Suppose, for a contradiction, that this is not true. Then there are integers α and β for which $(y_1 H)^\alpha = (y_2 H)^\beta \neq H$. The equality $(y_1 H)^\alpha = (y_2 H)^\beta$ implies that $y_1^\alpha y_2^{-\beta} \in H$ and so by the assumption on nonidentity elements we deduce that $y_1^\alpha = y_2^\beta$. By assumption $\langle y_1 \rangle \cap \langle y_2 \rangle = \{1\}$ and so $y_1^\alpha = y_2^\beta = 1$, contradicting the inequality $(y_1 H)^\alpha \neq H$. Therefore the subgroups $\langle y_1 H \rangle$ and $\langle y_2 H \rangle$ have trivial intersection as claimed. By Theorem 2.24 of [8], the subgroup generated by any two normal subgroups which intersect trivially is isomorphic to the (external) direct product of those subgroups, proving the case $r = 2$.

In the inductive step, we use the same argument to show that the groups $\langle y_1 H \rangle$ and $\langle y_2 H, y_3 H, \dots, y_r H \rangle$ have trivial intersection and therefore that $\langle y_1 H, y_2 H, \dots, y_r H \rangle \cong \langle y_1 H \rangle \times \langle y_2 H, y_3 H, \dots, y_r H \rangle$. The inductive hypothesis applied to the elements y_2, y_3, \dots, y_r then proves the Lemma. \blacksquare

We can now characterise the form of the quotient groups G/H_i and Q/H_i as discussed. We write $\prod_{u=1}^r \mathbb{Z}_{\alpha_u}$ for the direct product $\mathbb{Z}_{\alpha_1} \times \mathbb{Z}_{\alpha_2} \times \dots \times \mathbb{Z}_{\alpha_r}$.

LEMMA 7 *Let G be the group $\prod_{u=1}^{2r} \mathbb{Z}_{p^{1+\alpha_u}}$ containing a subgroup $Q \cong \mathbb{Z}_p^{2r}$, where p is prime and $\alpha_u \geq 0$. Let H_0, H_1, \dots, H_{p^r} be the subgroups of G of order p^r corresponding to hyperplanes when viewed as subgroups of Q . Then for each H_i there exists a r -element subset S of $\{1, 2, \dots, 2r\}$ such that $G/H_i \cong \prod_{u \notin S} \mathbb{Z}_{p^{1+\alpha_u}} \times \prod_{u \in S} \mathbb{Z}_{p^{\alpha_u}}$. Moreover, for each H_i a suitable choice of generators of G ensures that $Q/H_i \cong \mathbb{Z}_p^r$ is contained in the first r direct factors of G/H_i as specified. Furthermore if H_0 is contained in a subgroup of G isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^2}^{r-1}$ then, for each $H_i \neq H_0$, Q/H_i is contained in a subgroup of G/H_i isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^2}^{r-1}$.*

Proof. This result is given as Lemma 4.4 of [3], except for the final sentence in the case when H_0 is not contained in a subgroup of G isomorphic to $\mathbb{Z}_{p^2}^r$. To prove this case, let $\alpha_1 = 0$ and $\alpha_u \geq 1$ for $2 \leq u \leq r$ and let $\{x_u \mid 1 \leq u \leq 2r\}$ be a set of generators of G such that $G = \langle x_u \mid x_u^{p^{1+\alpha_u}} = 1 \rangle$ and $H_0 = \langle x_1, x_2^{p^{\alpha_2}}, \dots, x_r^{p^{\alpha_r}} \rangle$. Fix $H_i \neq H_0$ and put $y_1 = x_1$ and $y_u = x_u^{p^{\alpha_u}}$ for $2 \leq u \leq r$. Clearly $\langle y_u \rangle \cap \langle y_j \mid j \neq u, j \leq r \rangle = \{1\}$ for each u in the range $1 \leq u \leq r$. Since the hyperplanes H_0, H_1, \dots, H_{p^r} partition the nonidentity elements of Q and by assumption $H_0 = \langle y_1, y_2, \dots, y_r \rangle$, no nonidentity element of the form $\prod_{u=1}^r y_u^{j_u}$ (where $0 \leq j_u < p$) is contained in H_i . Applying Lemma 6 and then substituting for the y_u in terms of the x_u we find that $T = \langle x_1 H_i, x_2^{p^{\alpha_2}} H_i, \dots, x_r^{p^{\alpha_r}} H_i \rangle \cong \langle x_1 H_i \rangle \times \langle x_2^{p^{\alpha_2}} H_i \rangle \times \dots \times \langle x_r^{p^{\alpha_r}} H_i \rangle \cong \mathbb{Z}_p^r$. Since T is a subgroup of Q/H_i and has the same order p^r , it follows that $T = Q/H_i$.

Now $T = Q/H_i$ is contained in the subgroup $V = \langle x_1 H_i, x_2^{p^{\alpha_2-1}} H_i, \dots, x_r^{p^{\alpha_r-1}} H_i \rangle$. Put $z_1 = x_1$ and $z_u = x_u^{p^{\alpha_u-1}}$ for $2 \leq u \leq r$. We wish to apply Lemma 6 to z_1, z_2, \dots, z_r to

conclude that $V \cong \langle x_1 H_i \rangle \times \langle x_2^{p^{a_2-1}} H_i \rangle \times \cdots \times \langle x_r^{p^{a_r-1}} H_i \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}^{r-1}$ as required. We can do so by showing that if $z = \prod_{u=1}^r z_u^{j_u} \in H_i$ (where $0 \leq j_1 < p$ and $0 \leq j_u < p^2$ for $2 \leq u \leq r$) then $z = 1$. Now H_i is isomorphic to \mathbb{Z}_p^r and so $z^p = 1$. Writing this equation in terms of the y_u defined above we get $\prod_{u=2}^r y_u^{j_u} = 1$, which implies that $j_u = p j'_u$ for each u in the range $2 \leq u \leq r$ (where $0 \leq j'_u < p$). Therefore $z = z_1^{j_1} \prod_{u=2}^r (z_u^p)^{j'_u}$, and since $H_0 = \langle z_1, z_2^p, \dots, z_r^p \rangle$ we have shown that $z \in H_0 \cap H_i = \{1\}$. This completes the proof. ■

We shall apply Lemma 7 with $p = 2$ and $r = 3$ to reduce that inductive step of the proof of our main result to two possibilities, depending on whether the quotient group Q/H_i is contained in a subgroup isomorphic to \mathbb{Z}_4^3 or not (in which case it must be contained in a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2$). When Q/H_i is contained in a subgroup isomorphic to \mathbb{Z}_4^3 we shall make use of BSs whose existence is given by the following special case ($r = 3, i = 1$) of Corollary 7.9 of [3]:

THEOREM 8 *For each d and c satisfying $2 \leq c \leq d$, there exists a $(2^{3(d+c)-5}, 2^{3d-1}, 2^{3(d-c)+3})$ BS on any abelian group $G_{d,c}$ of order $2^{3(d+c)-2}$ and exponents at most 2^c relative to any subgroup $U_{d,c} \cong \mathbb{Z}_2^3$, where $U_{d,c}$ is contained in a subgroup of $G_{d,c}$ isomorphic to \mathbb{Z}_4^3 and where both of the following hold:*

- (i) *For $c = d$, $G_{d,c}/U_{d,c}$ contains a subgroup of index 2 and exponent at most 2^{d-1} .*
- (ii) *For $d > 2$ and $c = d - 1$, $G_{d,c}/U_{d,c}$ contains a subgroup of index 2^4 and exponent at most 2^{d-2} .*

Finally we require the following result on transferring BSs from a smaller group to a larger group, given as Lemma 2.1 in [3]:

LEMMA 9 *Suppose there exists a (a, \sqrt{at}, t) BS on an abelian group G relative to a subgroup U . Then there exists a $(as, \sqrt{at}, t/s)$ BS on G' relative to U , where s divides t and G' is any abelian group containing G as a subgroup of index s .*

We are now ready to state and prove the main result of the paper, namely the construction of a new family of BSs which leads to a new family of RDSs.

THEOREM 10 *There exists a $(8, 4, 2)$ BS on the group $\mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$ relative to the subgroup \mathbb{Z}_2^3 contained in the first three direct factors. There exists a $(2^6, 2^5, 2^4)$ BS on the group $\mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^4$ relative to the subgroup \mathbb{Z}_2^3 contained in the first three direct factors. For each d and c satisfying $2 \leq c \leq d$, there exists a $(3(d+c)-2, 2^{3d+2}, 2^{3(d-c)+6})$ BS on any abelian group $G_{d,c}$ of order $2^{3(d+c)+1}$ and exponent at most 2^c relative to any subgroup $U_{d,c} \cong \mathbb{Z}_2^3$, where $U_{d,c}$ is contained in a subgroup of $G_{d,c}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2$ but not in a subgroup isomorphic to \mathbb{Z}_4^3 and where, for $c = d$, $G_{d,c}/U_{d,c}$ contains a subgroup of index 2^4 and exponent at most 2^{d-1} .*

Proof. The required $(8, 4, 2)$ BS is given by Example 4. The required $(2^6, 2^5, 2^4)$ BS is given by Theorem 5 and Lemma 7. Then by Lemma 9 with $s = 2$, there exists a $(2^7, 2^5, 2^3)$

BS on both of the groups $\mathbb{Z}_2 \times \mathbb{Z}_4^3 \times \mathbb{Z}_2^3$ and $\mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^5$ relative to the subgroup \mathbb{Z}_2^3 contained in the first three direct factors.

We next establish the case $d = c = 2$ by showing there exists a $(2^{10}, 2^8, 2^6)$ BS on any group $G_{2,2}$ of order 2^{13} and exponent 4 relative to $U_{2,2} \cong \mathbb{Z}_2^3$, where $U_{2,2}$ is contained in a subgroup of $G_{2,2}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2$ but not in a subgroup isomorphic to \mathbb{Z}_4^3 . We shall apply Theorem 5, choosing the subgroup $Q_{2,2} \cong \mathbb{Z}_2^6$ of $G_{2,2}$ to contain $U_{2,2}$ and to contain direct factors \mathbb{Z}_4 to $G_{2,2}$ in preference to direct factors \mathbb{Z}_2 , and choosing the subgroups H_i of $G_{2,2}$ corresponding to hyperplanes of $Q_{2,2}$ so that $H_0 = U_{2,2}$. The required $(2^{10}, 2^8, 2^6)$ BS exists provided that, for each hyperplane $H_i \neq H_0$, there exists a $(2^7, 2^5, 2^3)$ BS on $G_{2,2}/H_i$ relative to $Q_{2,2}/H_i$. Now by Lemma 7, $G_{2,2}/H_i$ is isomorphic to one of the groups $\mathbb{Z}_2 \times \mathbb{Z}_4^3 \times \mathbb{Z}_2^3$, $\mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^5$, $\mathbb{Z}_4^4 \times \mathbb{Z}_2^2$ and $\mathbb{Z}_4^3 \times \mathbb{Z}_2^4$, with $Q_{2,2}/H_i$ contained in the first three direct factors of the group. For the first two groups the required $(2^7, 2^5, 2^3)$ BS is given in the preceding paragraph; for the second two groups it is given by the case $d = c = 2$ of Theorem 8.

The remainder of the proof is by induction on d . Assume the case $d - 1$ to be true (for each value of c in the range $2 \leq c \leq d - 1$). Let $U_{d,c}$ be contained in the first three direct factors of $G_{d,c}$ and order the remaining direct factors of $G_{d,c}$ in non-increasing order of their exponent. Choose $Q_{d,c} \cong \mathbb{Z}_2^6$ to be contained in the first six direct factors of $G_{d,c}$ and choose the subgroups H_i as above so that $H_0 = U_{d,c}$. By Theorem 5 it is sufficient to show, for each $H_i \neq H_0$, that there exists a $(2^{3(d+c)-5}, 2^{3d-1}, 2^{3(d-c)+3})$ BS on $G_{d,c}/H_i$ relative to $Q_{d,c}/H_i$. We distinguish two cases.

Case 1: Q/H_i is contained in a subgroup isomorphic to \mathbb{Z}_4^3 . In this case Theorem 8, using the same values for d and c , gives the required BS provided the associated conditions (i) and (ii) are met.

Condition (i) is that $(G_{d,d}/H_i)/(Q_{d,d}/H_i) \cong G_{d,d}/Q_{d,d}$ contains a subgroup of index 2 and exponent at most 2^{d-1} . Suppose, for a contradiction, that this is not the case. Since $G_{d,d}$ has exponent at most 2^d it follows that $G_{d,d}/Q_{d,d}$ contains a subgroup isomorphic to $\mathbb{Z}_{2^d}^2$. By the ordering of exponents of all but the first three direct factors of $G_{d,d}$ this implies that $G_{d,d}$ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{2^d}^5$. This contradicts the assumption that $G_{d,d}$ contains a subgroup of index 2^4 and exponent at most 2^{d-1} .

Condition (ii) is that, for $d > 2$, $G_{d,d-1}/Q_{d,d-1}$ contains a subgroup of index 2^4 and exponent at most 2^{d-2} . Supposing this not to be the case, it follows similarly that $G_{d,d-1}/Q_{d,d-1}$ contains a subgroup isomorphic to $\mathbb{Z}_{2^{d-1}}^5$ and therefore that $G_{d,d-1}$ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{2^{d-1}}^8$. But then the order of $G_{d,d-1}$ would exceed the stipulated value of 2^{6d-2} , giving a contradiction.

Case 2: Q/H_i is not contained in a subgroup isomorphic to \mathbb{Z}_4^3 . By Lemma 7, Q/H_i is therefore contained in a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2$.

For $c \leq d - 1$ we apply the inductive hypothesis, with the same value of c , to give the required BS provided the associated condition is met. For $c \leq d - 2$ there is no condition to check; for $c = d - 1$ the condition is that $(G_{d,d-1}/H_i)/(Q_{d,d-1}/H_i) \cong G_{d,d-1}/Q_{d,d-1}$ contains a subgroup of index 2^4 and exponent at most 2^{d-2} . The proof of this condition is identical to that given in Case 1 above.

For $c = d$ there is no inductive hypothesis with the value c to provide the required $(2^{6d-5}, 2^{3d-1}, 2^3)$ BS on $G_{d,d}/H_i$ relative to $Q_{d,d}/H_i$. Instead we shall use the inductive

hypothesis with the value $c = d - 1$, together with Lemma 9, in the following way. Firstly we claim that $G_{d,d}/Q_{d,d}$ contains a subgroup of index 2^7 and exponent at most 2^{d-2} . To show this, note that $G_{d,d}$ has exponent at most 2^d and by assumption contains a subgroup of index 2^4 and exponent at most 2^{d-1} , so that $G_{d,d}$ contains at most four direct factors \mathbb{Z}_{2^d} . By the ordering of exponents of all but the first three direct factors of $G_{d,d}$ this implies that $G_{d,d}/Q_{d,d}$ contains at most one direct factor \mathbb{Z}_{2^d} . Therefore, if the claim were false, $G_{d,d}/Q_{d,d}$ would contain a subgroup isomorphic to either $\mathbb{Z}_{2^d} \times \mathbb{Z}_{2^{d-1}}^6$ or $\mathbb{Z}_{2^{d-1}}^8$ and in either case the order of $G_{d,d}/Q_{d,d}$ would exceed its stipulated value of 2^{6d-5} ; this establishes the claim. Now it can be verified that the claim implies that $G_{d,d}/H_i$ contains a subgroup S/H_i (containing $Q_{d,d}/H_i$ in a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2$ but not in a subgroup isomorphic to \mathbb{Z}_4^3) of index 8 and exponent at most 2^{d-1} such that $(S/H_i)/(Q_{d,d}/H_i) \cong S/Q_{d,d}$ contains a subgroup of index 2^4 and exponent at most 2^{d-2} . (This is achieved by choosing a suitable subgroup $S/Q_{d,d}$ of $G_{d,d}/Q_{d,d}$ of index 8 for which the pre-image S/H_i of $(S/H_i)/(Q_{d,d}/H_i)$, under the quotient mapping from $G_{d,d}/H_i$ to $(G_{d,d}/H_i)/(Q_{d,d}/H_i)$, has exponent at most 2^{d-1} . For a detailed justification of a similar implication see the proof of Theorem 7.5 of [3].) Then the inductive hypothesis with the value $c = d - 1$ gives a $(2^{6d-8}, 2^{3d-1}, 2^6)$ BS on S/H_i relative to $Q_{d,d}/H_i$, and the required $(2^{6d-5}, 2^{3d-1}, 2^3)$ BS is obtained by applying Lemma 9 with $s = 8$. ■

Although each value of c in Theorem 10 gives rise, under Theorem 3, to semi-regular RDSs not occurring for any other value of c , we consider the small rank case $c = d$ to be of most interest and so state the resulting RDSs explicitly. (For clarity we have not stated the result of applying Theorem 3 to the $(8, 4, 2)$ and $(2^6, 2^5, 2^4)$ BSs of Theorem 10.)

COROLLARY 11 *For each $d \geq 2$, there exists a $(2^{6d+4}, 8, 2^{6d+4}, 2^{6d+1})$ semi-regular RDS in any abelian group G_d of order 2^{6d+7} relative to any subgroup $U_d \cong \mathbb{Z}_2^3$, where G_d contains a subgroup S_d of index 64 and exponent at most 2^d such that U_d is contained in a subgroup of S_d isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4^2$ but is not contained in a subgroup of S_d isomorphic to \mathbb{Z}_4^3 and such that S_d/U_d contains a subgroup of index 16 and exponent at most 2^{d-1} .*

The best previously known results for semi-regular RDSs of small rank having these parameters are those given by putting $r = 3$ and $j = 4$ in Corollary 8.4 of [3]. However Corollary 8.4 (ii) of [3] requires the rank of G_d to be at least 8 and Corollary 8.4 (v) of [3] requires U_d to be contained in a subgroup of G_d isomorphic to \mathbb{Z}_4^3 . Corollary 11 improves on both of these results, for example by establishing for each $d \geq 2$ the existence of a $(2^{6d+4}, 8, 2^{6d+4}, 2^{6d+1})$ semi-regular RDS in $G_d = \mathbb{Z}_2 \times \mathbb{Z}_{2^{d+1}}^6$ (having rank 7) relative to the subgroup $U_d \cong \mathbb{Z}_2^3$ contained in the first three direct factors.

This section illustrates that the discovery of a single new example of a semi-regular RDS can be used to construct recursively an infinite family of such RDSs using Theorems 5 and 3 (although the only new solutions to the square root problem in this paper are those given in the previous section).

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